Tutorial 5 for MATH 2020A (2024 Fall)

1. Let $R \subset \mathbb{R}^2$ be the region bounded by the lines $y = -\frac{3}{2}x + 1$, $y = -\frac{3}{2}x + 3$, $y = -\frac{1}{4}x$ and $y = -\frac{1}{4}x + 1$. For $f(x, y) = 3x^2 + 14xy + 8y^2$, evaluate the integral

$$\iint_R f(x, y)\,\mathrm{d}A.$$

> **Solution:** $\frac{64}{5}$

2. A thin plate of constant density covers the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > 0, b > 0$, in the $xy-$plane. Find the moment of inertia of the plate about the origin.

> **Solution:** $\frac{\pi}{4}ab(a^2 + b^2)$

3. Evaluate

$$\iiint_D |xyz|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

over the solid ellipsoid D,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

> **Solution:** $\frac{(abc)^2}{6}$

4. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path C_1 followed by C_2 from $(0, 0, 0)$ to $(1, 1, 1)$ given by

$$C_1 : \mathbf{r}(t) = (t, t^2, 0), t \text{ goes from 0 to 1,}$$
$$C_2 : \mathbf{r}(t) = (1, 1, t), t \text{ goes from 0 to 1.}$$

> **Solution:** $\frac{\sqrt{5}^3}{6} + \frac{3}{2}$

5. Let $C \subset \mathbb{R}^2$ be the closed curve $C_1 : \mathbf{r}(t) = (t, t^2)$, t goes from 0 to 1, followed by the curve $C_2 : \mathbf{r}(t) = (t, t)$, t goes from 1 to 0. Let $f(x, y) = x + \sqrt{y}$, evaluate

$$\int_C f(x, y)\,\mathrm{d}s.$$

Solution: *Correction:* I apologize for a serious mistake on the fifth tutorial, the answer of Q5 should be $\frac{\sqrt{5}^3-1}{6} + \frac{7\sqrt{2}}{6}$.

When calculating the integration on the curve C_2, **the following "formula"**

$$\int_{C_2} f \, \mathrm{d}s = \int_1^0 f(\mathbf{r}(t))|r'(t)| \, \mathrm{d}t \tag{1}$$

is incorrect!(One may deduce that (1) cannot be correct at least for C_2 in Q5 by noticing that the LHS of (1), as the limit of Riemann sum of a positive f, must be positive, while the RHS is negative, contradiction!)

In fact, the correct formula for integration over a curve C, say, $C : \mathbf{r}(t)$, t goes from a to b, where $a < b$, is given by

$$\int_{C} f \, \mathrm{d}s = \int_a^b f(\mathbf{r}(t))|r'(t)| \, \mathrm{d}t. \tag{2}$$

However, to apply this formula (2), it is crucial to make sure the parameter t is increasing! So one cannot apply this formula for C_2 in Q5 because the original parameter is decreasing.*(Actually, there is one such formula for the case parameter t is decreasing, which is basically add a minus sign in RHS of (1)).*

Thus to calculate $\int_{C_2} f \, \mathrm{d}s$, one shall realize that the integration of a function over a curve is independent of the choice of the parametrization of the curve, hence one may re-parametrize C_2 in Q5 by $\mathbf{r}(\tau) = (1-\tau, 1-\tau)$, τ goes from 0 to 1. This time the parameter τ is increasing, so one may apply the formula (2) to find

$$\begin{aligned}
\int_{C_2} f \, \mathrm{d}s &= \int_0^1 f(\mathbf{r}(\tau))|\mathbf{r}'(\tau)| \, \mathrm{d}\tau \\
&= \int_0^1 f(1-\tau, 1-\tau)\sqrt{2} \, \mathrm{d}\tau \\
&= \sqrt{2} \int_0^1 (1-\tau) + (1-\tau)^{\frac{1}{2}} \, \mathrm{d}\tau \\
&= \frac{7\sqrt{2}}{6}.
\end{aligned}$$